Section 3.2

Asymptotic Series, Laplace's Method,

Gamma Function, Stirling's Formula

3.2.0 Simplify probability function by Stirling's formula

- the exact form of $w(m, N)$ was obtained.
  m: position. *N*: total steps.

$$w(m,N) = \frac{C_p^N}{2^N} = \frac{N!}{2^N p!(N-p)!} \qquad p = (N+m)/2$$

- The calculation is tedious and heavy.
- James Stirling (*Scottish mathematician, 1692-1770*) presented a way to estimate it with enough accuracy.

Dominant term

$$\ln n! \sim \left(n+\frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi} + \frac{1}{12n} - \frac{1}{360n^3} + \cdots$$

$$n! \sim (2\pi n)^{1/2} n^n e^{-n}$$

Stirling's formula

will be proved later

3.2.0 Simplify probability function by Stirling's formula

$$w(m,N)=\frac{N!}{2^N p!(N-p)!}$$

$$p=\frac{N+m}{2}$$

$$\ln w(m,N)=\ln\frac{N!}{2^N p!(N-p)!}$$

$$\ln n! \sim \frac{1}{2}\ln 2\pi n + n\ln n - n$$

$$=\ln N! - \ln p! - \ln(N-p)! - N\ln 2$$

$$=\ln\sqrt{\frac{2}{\pi N}}-\frac{1+N}{2}\ln\left[1-\left(\frac{m}{N}\right)^2\right]+\frac{m}{2}\ln\left[1-\frac{m}{N}\right]-\frac{m}{2}\ln\left[1+\frac{m}{N}\right]$$

$$w(m,N)=\sqrt{\frac{2}{\pi N}}\left[1-\left(\frac{m}{N}\right)^2\right]^{-\frac{1+N}{2}}\left[1-\left(\frac{m}{N}\right)\right]^{\frac{m}{2}}\left[1+\left(\frac{m}{N}\right)\right]^{-\frac{m}{2}}$$

3.2.0 Simplify probability function by Stirling's formula

$$w(m,N)=\frac{N!}{2^N p!(N-p)!}$$

$$p=\frac{N+m}{2}$$

$$\ln n! \sim \frac{1}{2}\ln 2\pi n + n\ln n - n$$

$$\ln w(m,N)=\ln\frac{N!}{2^N p!(N-p)!}$$

$$=\ln N! - \ln p! - \ln(N-p)! - N\ln 2$$

$$=\frac{1}{2}\ln 2\pi+\frac{1}{2}\ln N+N\ln N-N \quad -\frac{1}{2}\ln 2\pi-\frac{1}{2}\ln p-p\ln p+p$$

$$-\frac{1}{2}\ln 2\pi-\frac{1}{2}\ln(N-p)-(N-p)\ln(N-p)+(N-p)\ -N\ln 2$$

$$=\frac{1}{2}\ln 2\pi+\frac{1}{2}\ln N+N\ln N-N \quad -\frac{1}{2}\ln 2\pi-\frac{1}{2}\ln\frac{N+m}{2}-\frac{N+m}{2}\ln\frac{N+m}{2}+\frac{N+m}{2}$$

$$-\frac{1}{2}\ln 2\pi-\frac{1}{2}\ln\frac{N-m}{2}-\frac{N-m}{2}\ln\frac{N-m}{2}+\frac{N-m}{2}\ -N\ln 2$$

$$=\frac{1}{2}\ln N+N\ln N-\frac{1}{2}\ln\frac{N+m}{2}-\frac{N+m}{2}\ln\frac{N+m}{2}-\frac{1}{2}\ln\frac{N-m}{2}-\frac{N-m}{2}\ln\frac{N-m}{2}\ -N\ln 2-\frac{1}{2}\ln 2\pi$$

$$=\frac{1}{2}\ln N+N\ln N-\frac{1}{2}\ln\frac{N^2-m^2}{4}-\frac{N}{2}\ln\frac{N^2-m^2}{4}+\frac{m}{2}\ln\frac{N-m}{N+m}-N\ln 2-\frac{1}{2}\ln 2\pi$$

$$=\frac{1}{2}\ln N+N\ln N-\frac{1+N}{2}\ln\frac{1-(m/N)^2}{4}N^2\ +\frac{m}{2}\ln\frac{1-(m/N)}{1+(m/N)}-N\ln 2-\frac{1}{2}\ln 2\pi$$

$$=\frac{1}{2}\ln N+N\ln N-N\ln 2-\frac{1+N}{2}\ln\frac{N^2}{4}-\frac{1}{2}\ln 2\pi-\frac{1+N}{2}\ln\left(1-(m/N)^2\right)\ +\frac{m}{2}\ln\left(1-(m/N)\right)-\frac{m}{2}\ln\left(1+(m/N)\right)$$

$$=\ln\sqrt{\frac{2}{\pi N}}-\frac{1+N}{2}\ln\left(1-(m/N)^2\right)\ +\frac{m}{2}\ln\left(1-(m/N)\right)-\frac{m}{2}\ln\left(1+(m/N)\right)$$

3.2.0 Simplify probability function by Stirling's formula

$$w(m,N)=\sqrt{\frac{2}{\pi N}}\left[1-\left(\frac{m}{N}\right)^2\right]^{-\frac{1+N}{2}}\left[1-\left(\frac{m}{N}\right)\right]^{\frac{m}{2}}\left[1+\left(\frac{m}{N}\right)\right]^{-\frac{m}{2}}$$

$$\xrightarrow{N\to\infty}$$

$$=\sqrt{\frac{2}{\pi N}}\ \exp\left[\left(\frac{m}{N}\right)^2\frac{1+N}{2}\right]\ \exp\left[-\left(\frac{m}{N}\right)\frac{m}{2}\right]\ \exp\left[-\left(\frac{m}{N}\right)\frac{m}{2}\right]$$

$$=\sqrt{\frac{2}{\pi N}}\exp\left[-\frac{(N-1)m^2}{2N^2}\right]$$

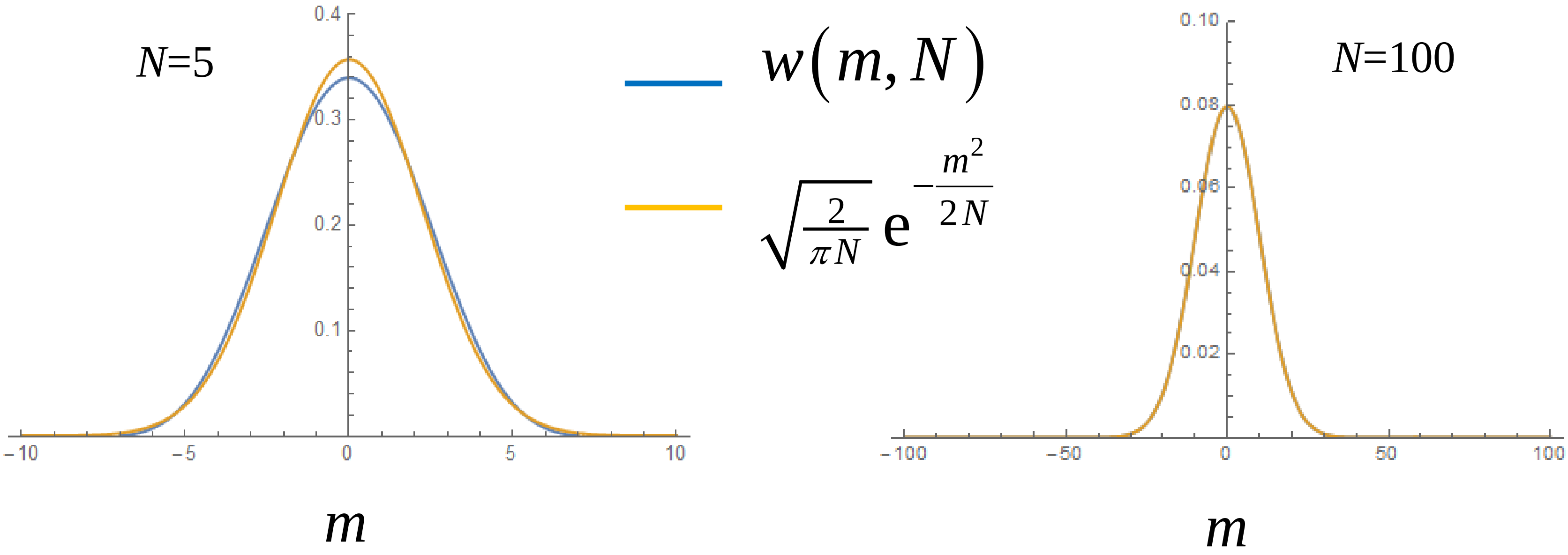
$$\approx\sqrt{\frac{2}{\pi N}}\exp\left(-\frac{m^2}{2N}\right)$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

$$w(m,N)\approx\sqrt{\frac{2}{\pi N}}\exp\left(-\frac{m^2}{2N}\right)$$

3.2.0 Simplify probability function by Stirling's formula

$$w(m,N) \approx \sqrt{\frac{2}{\pi N}} \exp\left(-\frac{m^2}{2N}\right)$$



Stirling's formula

- 2 heuristic & 8 rigorous approaches to derive

In this book, via gamma function

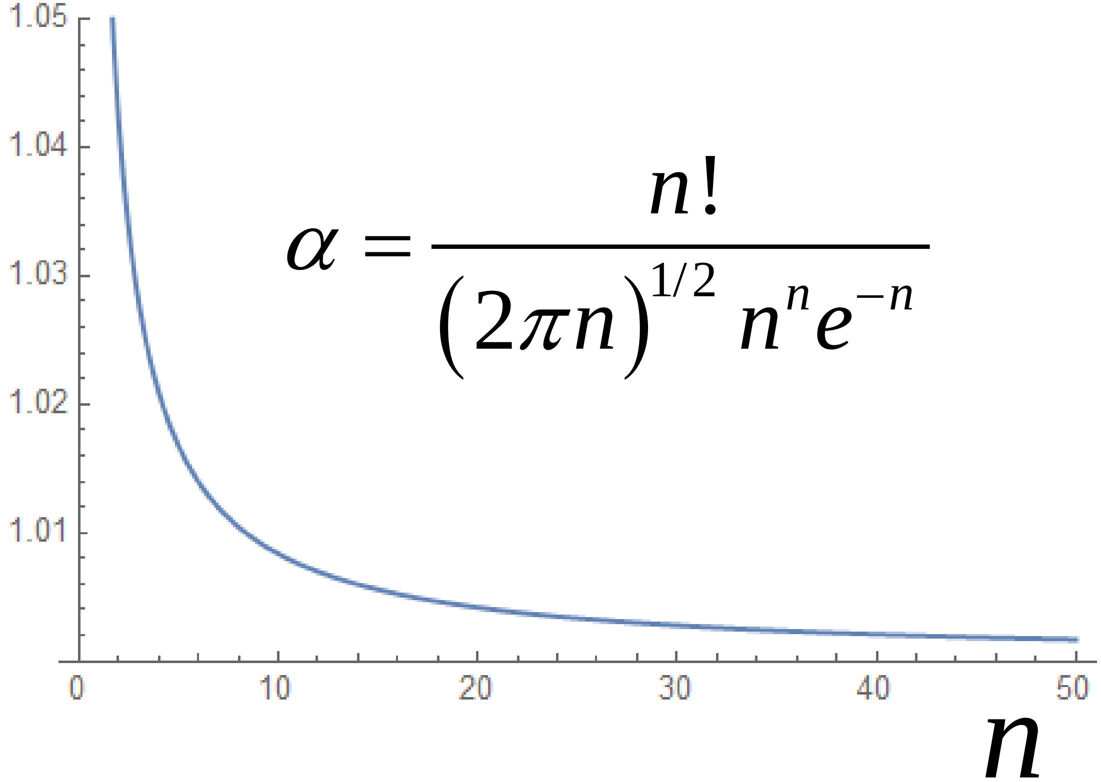
$$\ln n! \sim \left(n+\frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \cdots$$

- Diverge for any value of *n*.
- Not a series in rigorous mathematical sense.

3.2.0 Simplify probability function by Stirling's formula

- But ! The dominant term works!

$$n! \sim (2\pi n)^{1/2}\, n^n e^{-n}$$



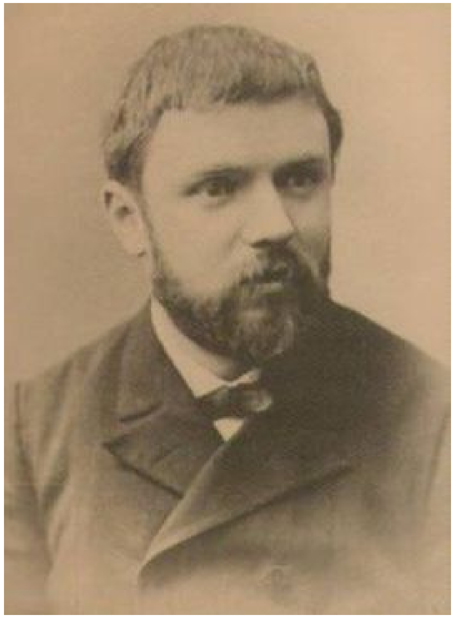
$$\alpha = \frac{n!}{(2\pi n)^{1/2}\, n^n e^{-n}}$$

n	α
5	1.01678
10	1.00837
50	1.00167
100	1.000837

Because it is an asymptotic series.

Jules Henri Poincaré introduced asymptotic expansion in 1886. This concept enables one to

- manipulate a large class of divergent series
- obtain numerical as well as qualitative results for many problems.



Example -1: ODE

$$\frac{dy}{dx} + y = \frac{1}{x}$$ for large x.

we have a solution in the form

$$y = \frac{1}{x} + \frac{2!}{x^2} + \frac{3!}{x^3} + \cdots \frac{(n-1)!}{x^n} + \cdots$$

this divergent series is useful for numerical calculations, and called an asymptotic series

Example-2: regular quadratic

$$x^2 + \varepsilon x - 1 = 0$$

ε : a small constant, say ε =0.0000001.

Exact solutions $x = \dfrac{-\varepsilon \pm \sqrt{\varepsilon^2 + 4}}{2}$

As ε=0, we have the unperturbed solution

$$x^2 + \cancel{\varepsilon} x - 1 = 0 \Rightarrow \quad x = \pm 1$$

Taylor expansion of the exact solution:

$$x = \begin{cases} 1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} - \frac{\varepsilon^4}{128} + O\left(\varepsilon^6\right) \\ -1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \frac{\varepsilon^4}{128} + O\left(\varepsilon^6\right) \end{cases}$$

converge if

$$|\varepsilon| < 2$$

Series method

power series expansion around $x = \pm 1$

$$x = \pm 1 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots$$

The same Taylor expansions can be reproduced.

Example-3: singular quadratic

$$\epsilon x^2 + x - 1 = 0$$

ε is a small constant, say ε =0.0000001.

Exact solutions $$x = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}$$

As ε=0, we have the unperturbed solution

$$\cancel{\epsilon x^2} + x - 1 = 0 \quad \Rightarrow \quad x = 1$$

Only one root !

Taylor expansion of the exact solution:

$$x = \begin{cases} 1-\varepsilon+2\varepsilon^2-5\varepsilon^3+O\left(\varepsilon^4\right) \\ -\frac{1}{\varepsilon}-1+\varepsilon-2\varepsilon^2+5\varepsilon^3+O\left(\varepsilon^4\right) \end{cases}$$

converge if $|\varepsilon|<1/4$

Expansion method

Assuming power series

Why? be patient

$$x = \frac{a_{-1}}{\varepsilon}+a_0+a_1\varepsilon+a_2\varepsilon^2+\cdots$$

Substituting into $\epsilon x^2 + x - 1 = 0$

$$\varepsilon^{-1}\left(a_{-1}^2+a_{-1}\right)+\varepsilon^0\left(2a_{-1}a_0+a_0-1\right)+\varepsilon\left(2a_{-1}a_1+a_0^2+a_1\right)$$
$$+\cdots=0$$

Comparing coefficients of ε of same order

ε^{-1} : $a_{-1}^2+a_{-1}=0$ $\quad a_{-1}=-1$ $\quad$ or $\quad a_{-1}=0$

ε^{0} : $2a_{-1}a_0+a_0-1=0$ $\quad a_0=-1$ $\quad a_0=1$

ε^{1} : $2a_{-1}a_1+a_0^2+a_1=0$ $\quad a_1=1$ $\quad a_1=-1$

$$x=-\frac{1}{\varepsilon}-1+\varepsilon+\cdots$$

Singular root

$$x=1-\varepsilon$$
$$+2\varepsilon^2\cdots$$

Regular root

Rescaling method

balance the three terms $\varepsilon x^2 + x - 1 = 0$

(1) x and -1 is comparable, assuming εx^2 is smaller than other two terms.

$$x \sim O(1) \qquad \varepsilon x^2 \sim o(1)$$

$$\varepsilon x^2 + x - 1 = 0 \Rightarrow x \approx 1$$

Good guess

(2) εx^2 and -1 is comparable, assuming x is smaller than other two terms.

$$\varepsilon x^2 \sim O(1) \qquad x \sim o(1)$$

$$\varepsilon x^2 + x - 1 = 0 \Rightarrow |x| \approx \frac{1}{\sqrt{\varepsilon}} >> 1$$

Bad guess

(3) εx^2 and x is comparable, assuming both terms >> 1.

$$\varepsilon x^2 \sim O(x) \gg 1$$

$$\varepsilon x^2 + x - 1 = 0 \Rightarrow |x| \approx \frac{1}{\varepsilon} \gg 1$$

Self consistent

When εx^2 and x balance, x is very larger $x \sim O(\varepsilon^{-1})$

rescaling x, $x = \dfrac{X}{\varepsilon}$ with $X \sim O(1)$

We get a regular looking problem

$$X^2 + X - \varepsilon = 0$$

Using regular expansion

$$X = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots$$

Comparing coefficients of ε of same order, we get

$$x = -\frac{1}{\varepsilon} - 1 + \varepsilon + \cdots$$

Example-4: Asymptotic series by parts integration

Error function $$\mathrm{Erf}\left(z\right)=\frac{2}{\sqrt{\pi}}\int_{0}^{z}e^{-t^{2}}dt$$

- In the 19th century, error function from the theory of errors appeared in several contexts unrelated to probability, e.g. refraction and heat conduction.
- In 1871, J. W. Glaisher wrote that "Erf(*x*) may fairly claim at present to rank in importance next to the trigonometrical and logarithmic functions."
- Glaisher introduced the symbol Erf and the name it *error function*.

$$\mathrm{Erf}\left(z\right)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(-1\right)^n z^{2n+1}}{\left(2n+1\right)n!}$$

$$=\frac{2}{\sqrt{\pi}}\left(z-\frac{z^3}{3}+\frac{z^5}{10}-\frac{z^7}{42}+\frac{z^9}{216}-\frac{z^{11}}{1320}+\cdots\right)$$

D'Alembert's ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\left(2n+1\right)n!\mathrm{z}^{2n+3}}{\left(2n+3\right)\left(n+1\right)!\mathrm{z}^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{\left(2n+1\right)z^2}{\left(2n+3\right)\left(n+1\right)}\right|=0$$

This series is convergence.

Consider an alternative form of the error function

$$\mathrm{Erf}\left(z\right)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt=\frac{2}{\sqrt{\pi}}\left(\int_0^\infty+\int_\infty^z\right)e^{-t^2}dt$$

$$=1-\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt=1-\mathrm{Erfc}\left(z\right)$$

← complementary error function

Integrating by parts

$$\int udv=uv-\int vdu$$

$$\frac{\mathrm{Erfc}\left(z\right)}{2/\sqrt{\pi}}=\int_z^\infty e^{-t^2}dt=-\int_z^\infty\frac{de^{-t^2}}{2t}=\frac{e^{-z^2}}{2z}-\int_z^\infty\frac{e^{-t^2}}{2t^2}dt$$

and 3 more times

$$\frac{\mathrm{Erfc}\left(z\right)}{2/\sqrt{\pi}}=\frac{e^{-z^2}}{2z}\left(1-\frac{1}{2z^2}+\frac{1\cdot3}{\left(2z^2\right)^2}-\frac{1\cdot3\cdot5}{\left(2z^2\right)^3}+\frac{1\cdot3\cdot5\cdot7}{\left(2z^2\right)^4}\cdots\right)+\int_z^\infty\frac{105e^{-t^2}}{16t^8}dt$$

3.2.1 Examples of asymptotics

the remainder can be bounded by

$$\left|R_5\right| = \int_z^\infty \frac{105 e^{-t^2}}{16 t^8} dt = \int_z^\infty \frac{105 d e^{-t^2}}{32 t^9} < \frac{105}{32 z^9} \int_z^\infty d e^{-t^2} = \frac{105 e^{-z^2}}{32 z^9} = a_5$$

$$\boxed{\left|R_5\right| < a_5}$$

double factorial

$$n!! \equiv \begin{cases} n \cdot (n-2) \ldots 5 \cdot 3 \cdot 1 & n > 0 \text{ odd} \\ n \cdot (n-2) \ldots 6 \cdot 4 \cdot 2 & n > 0 \text{ even} \\ 1 & n = -1, 0. \end{cases}$$

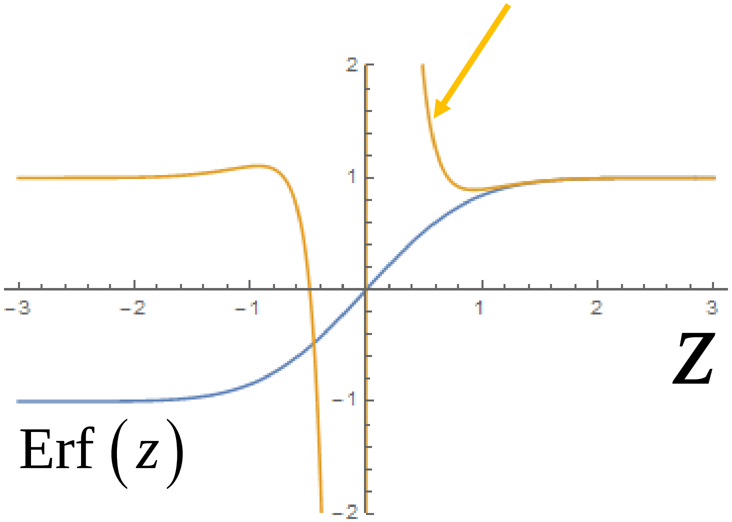
Thus we have proven that as $z \longrightarrow \infty$

$$\operatorname{Erf}(z) = 1 - \operatorname{Erfc}(z) \approx 1 - \frac{e^{-t^2}}{z\sqrt{\pi}} \left(1 + \sum_{n=2}^{N} (-1)^{N+1} \frac{(2N-3)!!}{\left(2z^2\right)^{N-1}} \right) + O\left(z^{-2N+1}\right)$$

- This expansion for the error function diverges
- However, the truncated series, is useful

3.2.1 Examples of asymptotics

3 terms $1 - \dfrac{e^{-z^2}}{z\sqrt{\pi}}\left(1 - \dfrac{1}{2z^2}\right)$



$z = 2.5$	$z = 3$
$\mathrm{Erf}(z) = 0.999593$	$\mathrm{Erf}(z) = 0.999978$
3 terms give 0.999599	3 terms give 0.9999778

This expansion has some important properties:

- the leading term is roughly correct
- further terms are corrections of decreasing size.

This property is called asymptoticness.

Example-5: from our book

To evaluate the integral

$$f(x) = \int_x^\infty t^{-1} e^{x-t} dt \qquad \text{as } x >> 1$$

Notice the incomplete gamma function reads

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$$

$$\boxed{f(x) = e^x \int_x^\infty t^{-1} e^{-t} dt = e^x \Gamma(0, x)}$$

integration by parts,

$$\int u\,dv = uv - \int v\,du$$

$$f(x) = \int_x^\infty t^{-1}e^{x-t}dt = -\int_x^\infty t^{-1}de^{x-t} = -t^{-1}e^{x-t}\Big|_x^\infty + \int_x^\infty e^{x-t}t^{-2}dt$$

$$= \frac{1}{x} + \int_x^\infty e^{x-t}t^{-2}dt = \frac{1}{x} - \frac{1}{x^2} + 2\int_x^\infty e^{x-t}t^{-3}dt$$

by successive integration by parts,

$$\boxed{f(x) = S_n(x) + R_n(x)}$$

where $S_n(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}$

$$R_n(x) = (-1)^n n! \int_x^\infty t^{-(n+1)}e^{x-t}dt = (-1)^n n! e^x \Gamma(-n, x)$$

We have some observations

- as $n \to \infty$, $S_n(x)$ diverges for a fixed x.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n!x^n}{(n-1)!x^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{x}\right| = \infty$$

- as $x \to \infty$, $R_n(x) \to 0$, for a fixed n.

$$\left|R_n(x)\right| < n!x^{-(n+1)}\int_x^\infty e^{x-t}dt = n!x^{-(n+1)} = \left|a_{n+1}\right|$$

- The ratio of the remainder to the last term approaches zero, as $x \to \infty$

$$\left|\frac{R_n(x)}{a_n(x)}\right| < \left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{n!x^{-(n+1)}}{(n-1)!x^{-n}} = \frac{n}{x} < 1$$

- As $x \geq 2n$

$$|R_n(x)| < |a_{n+1}(x)| = n!x^{-(n+1)} < n!(2n)^{-(n+1)}$$

$$= \frac{1}{2^{n+1}n} \cdot \frac{n!}{n^n} = \frac{1}{2^{n+1}n} \cdot \left[\frac{n}{n} \frac{n-1}{n} \cdots \frac{1}{n} \right] < \frac{1}{2^{n+1}n^2}$$

which is small even for moderate values of *n.*

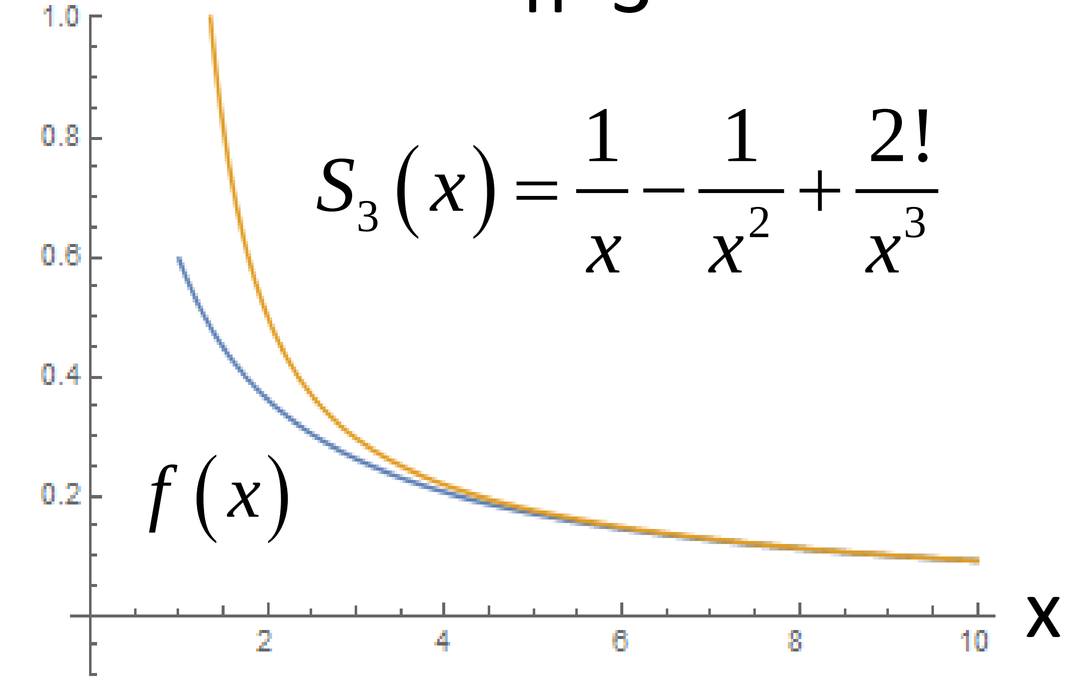
for example, as n=3, $x = 2n \geq 6$

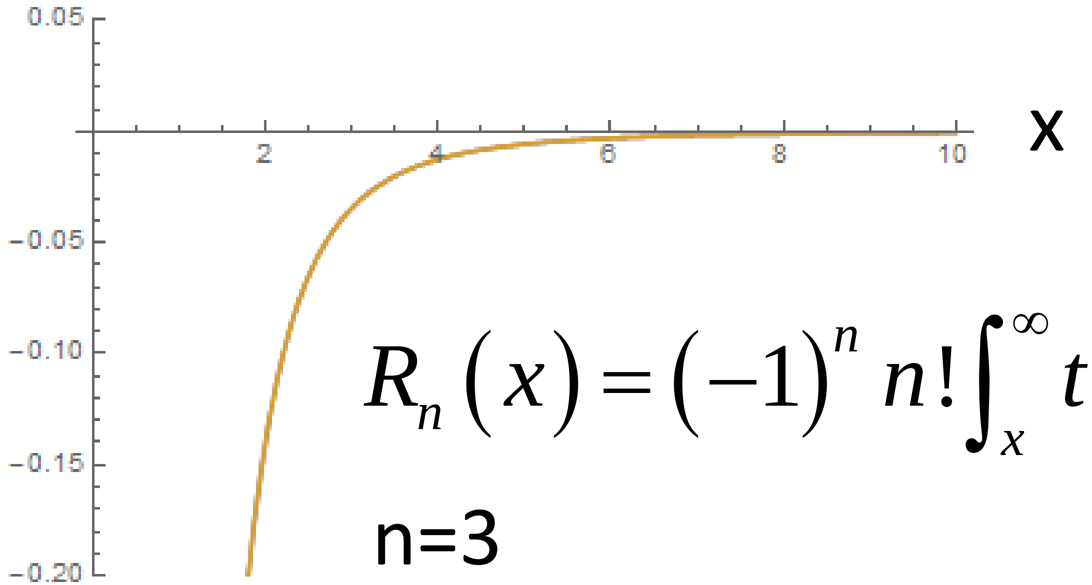
$$|R_n(x)| < \frac{1}{2^4 \, 3^2} = \frac{1}{144} \approx 0.007$$

3.2.1 Examples of asymptotics

n=3

$$S_3(x)=\frac{1}{x}-\frac{1}{x^2}+\frac{2!}{x^3}$$





$$R_n(x)=(-1)^n\, n!\int_x^\infty t^{-(n+1)}e^{x-t}\,dt$$

n=3

- As $x \geq 2n$

$$\left|R_n(x)\right| < |a_{n+1}(x)| = n!x^{-(n+1)} < n!(2n)^{-(n+1)}$$

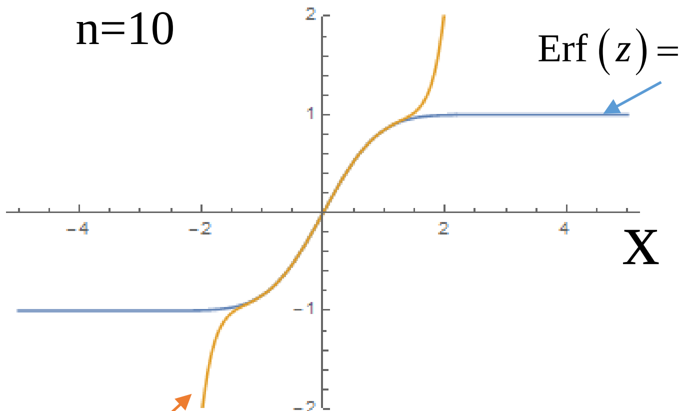
$$= \frac{1}{2^{n+1}n} \cdot \frac{n!}{n^n} = \frac{1}{2^{n+1}n} \cdot \left[\frac{n}{n}\frac{n-1}{n}\cdots\frac{1}{n}\right] < \frac{1}{2^{n+1}n^2}$$

which is small even for moderate values of *n.*

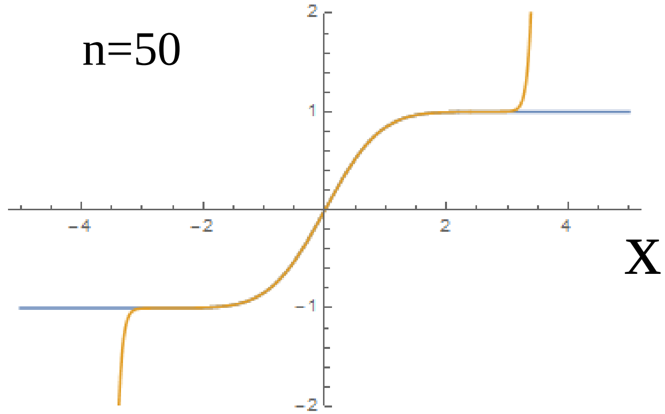
for example, as n=3, $x = 2n \geq 6$

$$\left|R_n(x)\right| < \frac{1}{2^4\,3^2} = \frac{1}{144} \approx 0.007$$

3.2.1 Examples of asymptotics

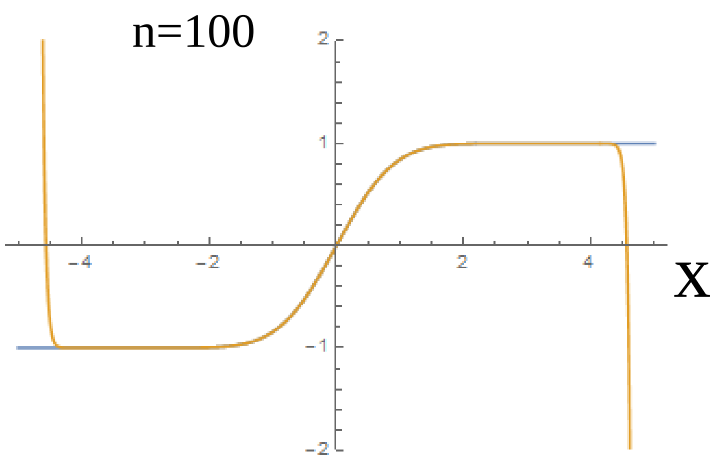


$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$



$$\frac{2x}{\sqrt{\pi}} - \frac{2x^3}{3\sqrt{\pi}} + \frac{x^5}{5\sqrt{\pi}} - \frac{x^7}{21\sqrt{\pi}} + \frac{x^9}{108\sqrt{\pi}} + O[x]^{11}$$

$$\frac{2x}{\sqrt{\pi}} - \frac{2x^3}{3\sqrt{\pi}} + \frac{x^5}{5\sqrt{\pi}} - \frac{x^7}{21\sqrt{\pi}} + \frac{x^9}{108\sqrt{\pi}} - \frac{x^{11}}{660\sqrt{\pi}} + \frac{x^{13}}{4680\sqrt{\pi}} - \frac{x^{15}}{37800\sqrt{\pi}} + \frac{x^{17}}{342720\sqrt{\pi}} - \frac{x^{19}}{3447360\sqrt{\pi}} + \frac{x^{21}}{38102400\sqrt{\pi}} - \frac{x^{23}}{459043200\sqrt{\pi}} + \frac{x^{25}}{5987520000\sqrt{\pi}} - \frac{x^{27}}{84064780800\sqrt{\pi}} + \frac{x^{29}}{1264085222400\sqrt{\pi}} - \frac{x^{31}}{20268952704000\sqrt{\pi}} + \frac{x^{33}}{345226033152000\sqrt{\pi}} - \frac{x^{35}}{6224529991680000\sqrt{\pi}} + \frac{x^{37}}{118443913555968000\sqrt{\pi}} - \frac{x^{39}}{2372079457972224000\sqrt{\pi}} + \frac{x^{41}}{49874491167621120000\sqrt{\pi}} - \frac{x^{43}}{1098455256691752960000\sqrt{\pi}} + \frac{x^{45}}{25290016374996172800000\sqrt{\pi}} - \frac{x^{47}}{607522393363796951040000\sqrt{\pi}} + \frac{x^{49}}{15200985842464366264320000\sqrt{\pi}} + O[x]^{51}$$



This convergent series is less useful in practice.

More asymptotic series

0-order Bessel functions

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2}$$

$$= 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 - \cdots$$

$$J_0(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \left[\left(1 - \frac{(3!!)^2}{2!(8x)^2} + \frac{(7!!)^2}{4!(8x)^4} - \cdots\right) \cos\left(x + \frac{1}{4}\pi\right)\right.$$

$$\left. + \left(\frac{1}{8x} - \frac{(5!!)^2}{3!(8x)^3} + \frac{(9!!)^2}{5!(8x)^5} - \cdots\right) \sin\left(x - \frac{1}{4}\pi\right)\right]$$

More asymptotic series

Laplace integral

$$T = \int_a^b g(t) e^{xh(t)} at \qquad I \sim g(a) e^{xh(a)} \sqrt{\frac{-\pi}{2xh''(a)}}$$

ODE

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + (\lambda^2 q_0 + q_2) y = 0$$

$$y \sim C_1 \frac{1}{(q_0 p)^{\frac{1}{4}}} \cos\left[\lambda \int_{x_0}^{x} \left(\frac{q_0}{p}\right)^{\frac{1}{2}} dx\right] + C_2 \frac{1}{(q_0 p)^{\frac{1}{4}}} \sin\left[\lambda \int_{x_0}^{x} \left(\frac{q_0}{p}\right)^{\frac{1}{2}} dx\right]$$

Consider function *f*(ε) , there are three possibilities

$$f(\varepsilon) = 0, \infty, A \qquad \text{as} \;\; \varepsilon \to 0, \quad 0 < A < \infty$$

- the speed at which *f*(ε) →∞ or *f*(ε) →0 can be expressed by comparing *gauge functions*.
- *gauge functions* may be

$$1, \varepsilon^{\pm n}, \varepsilon^{\pm 1/n}, \log \varepsilon^{-1}, \log\left(\log \varepsilon^{-1}\right), e^{\varepsilon^{-1}}, \sin \varepsilon, \sinh \varepsilon, \cdots$$

e.g.

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varepsilon^{1/2}} = 0$$

This indicates $f(\varepsilon)$ goes to 0 in a speed faster than that of $\varepsilon^{1/2}$

The Symbol O

Definition: if there exists a positive constant M such that $|\phi(x)| \leq M\,|\varphi(x)|$, as $x \to x_0$, we say

$$\phi(x) = O\left[\varphi(x)\right] \quad \text{or} \quad \phi(x) \sim \varphi(x)$$

ϕ is big-oh of φ — ϕ is asymptotic to φ

Meaning ϕ is of the same order of φ, or

$$\lim_{x \to x_0} \frac{\phi(x)}{\varphi(x)} = C$$

C is a constant

e.g. as x → 0 $\cos x = O(1)$, $x \sin x^2 = O\left(x^3\right)$

The Symbol o

Definition: if for every positive constant M such that $|\phi(x)| \le M\,|\varphi(x)|$, as $x \to x_0$, we say

$$\phi(x) = o\left[\varphi(x)\right] \qquad \text{or} \qquad \phi(x) << \varphi(x)$$

ϕ is small-oh of φ $\qquad$ ϕ is much less than φ

Meaning ϕ is of lower order of φ, or

$$\lim_{x \to x_0} \frac{\phi(x)}{\varphi(x)} = 0$$

e.g., as x → 0 $\quad \sin x = o(1), \quad \sin x^2 = o(x)$

e.g., as x → ∞ & any $a > 0$

$$e^{-x} = o\left(x^{-a}\right), \; x^a = o\left(e^x\right), \; \ln x = o\left(x^a\right), \; \ln\ln x = o(\ln x)$$

$$w(m,N)=\frac{N!}{2^N p!(N-p)!} \Rightarrow \sqrt{\frac{2}{\pi N}}\exp\left(-\frac{m^2}{2N}\right)$$

Stirling's formula $\ln n! \sim \left(n+\frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi} + \frac{1}{12n} - \frac{1}{360n^3} + \cdots$

$$x^2+\varepsilon x-1=0 \qquad x=\begin{cases} 1-\frac{\varepsilon}{2}+\frac{\varepsilon^2}{8}-\frac{\varepsilon^4}{128}+O\left(\varepsilon^6\right) \\ -1-\frac{\varepsilon}{2}-\frac{\varepsilon^2}{8}+\frac{\varepsilon^4}{128}+O\left(\varepsilon^6\right) \end{cases}$$

$$\epsilon x^2+x-1=0 \qquad x=\begin{cases} 1-\varepsilon+2\varepsilon^2-5\varepsilon^3+O\left(\varepsilon^4\right) \\ -\frac{1}{\varepsilon}-1+\varepsilon-2\varepsilon^2+5\varepsilon^3+O\left(\varepsilon^4\right) \end{cases}$$

convergent series

$$\mathrm{Erf}\left(z\right)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\left(-1\right)^n z^{2n+1}}{\left(2n+1\right)n!}$$

$$\mathrm{Erf}\left(z\right)=1-\mathrm{Erfc}\left(z\right)$$

asymptotic series

$$=1-\frac{e^{-t^2}}{z\sqrt{\pi}}\left(1+\sum_{n=2}^{N}\left(-1\right)^{N+1}\frac{\left(2N-3\right)!!}{\left(2z^2\right)^{N-1}}\right)+O\left(z^{-2N+1}\right)$$

$$f\left(x\right)=S_n\left(x\right)+R_n\left(x\right)$$

- as $n\rightarrow\infty$, $S_n(x)$ diverges for a fixed x.
- as $x\rightarrow\infty$, $R_n(x)\rightarrow 0$, for a fixed n.
- as $x\rightarrow\infty$, $|R_n| < |a_n|$

Review

$$\lim_{x\to x_0}\frac{\phi(x)}{\varphi(x)}=0 \qquad \phi(x)=o\left[\varphi(x)\right] \quad \text{or} \quad \phi(x)<<\varphi(x)$$

$$\lim_{x\to x_0}\frac{\phi(x)}{\varphi(x)}=C \qquad \phi(x)=O\left[\varphi(x)\right] \quad \text{or} \quad \phi(x)\sim\varphi(x)$$

- One can use a general sequence of gauge functions $\{\phi_n(\varepsilon)\}$ as *asymptotic sequence* as $\varepsilon \to 0$

e.g. $\varphi_n(\varepsilon) = \varepsilon^n, \quad (\log \varepsilon)^{-n}, \quad (\sin \varepsilon)^n \ldots$

- in the example of $\varepsilon x^2 + x - 1 = 0$, we see

$$x = 1 - \varepsilon + 2\varepsilon^2 + o(\varepsilon^2)$$

with asymptotic sequence $\{1, \varepsilon, \varepsilon^2, \ldots\}$

$$x = -\varepsilon^{-1} - 1 + \varepsilon - 2\varepsilon^2 + o(\varepsilon^2)$$

with asymptotic sequence $\{\varepsilon^{-1}, 1, \varepsilon, \varepsilon^2, \ldots\}$

3.2.2 Definition asymptotic expansion

- In terms of asymptotic sequences $\{\varphi_n(\varepsilon)\}$, we can expand functions $f(\varepsilon)$ in *asymptotic expansion* if constants a_n exist,

$$f(\varepsilon) = \sum_{n=0}^{N} a_n \phi_n(\varepsilon) + o\left[\phi_n(\varepsilon)\right] \qquad \text{as} \quad \varepsilon \to 0$$

or,

$$f(\varepsilon) \sim \sum_{n=0}^{N} a_n \phi_n(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0$$

- Usually, a function may depend on x and a small parameter ε. we may look for an expansion in the form

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x) \phi_n(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0$$

3.2.2 Definition asymptotic expansion

Consider an expansion

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \frac{A_{n+1}}{x^{n+1}} + \cdots \equiv S_n(x) + \frac{A_{n+1}}{x^{n+1}} + \cdots$$

$S_n(x)$ is an asymptotic expansion of f(x) , if

$$\lim_{x\to\infty} \frac{f(x) - S_n(x)}{A_n x^{-n}} = 0 \quad \text{with fixed } n$$

$$|R_n| < |a_n|$$

or

$$f(x) - S_n(x) = o\left(x^{-n}\right) \quad \text{with } x \to \infty, \text{ fixed } n$$

we then say

$$f(x) \sim \sum_{i=0}^{n} A_i x^{-i}, \quad x \to \infty$$

asymptotic to

If for any n, we always have

$$f(x) \sim \sum_{i=0}^{n} A_i x^{-i}, \quad x \to \infty$$

then we say *f*(*x*) has an asymptotic power series

$$f(x) \sim \sum_{i=0}^{\infty} A_i x^{-i}, \quad x \to \infty$$

This asymptotic series is usually divergent for any fixed *x*.

Convergence

An expansion $\sum_N f_n(x)$ is said to converge at a fixed value of *x* if given an arbitrary ε > 0 it is possible to find a number $N_0(x, \varepsilon)$ such that

$$\left| \sum_{M}^{N} f_n(x) \right| < \varepsilon \qquad \text{for } M, N > N_0$$

This property of convergence is less useful in practice.

Asymptoticness

The sum $\sum^{N} f_n(\varepsilon)$ is said to be an *asymptotic approximation* to $f(\varepsilon)$ as $\varepsilon \to 0$, if for each $n \le N$

$$\frac{R_n(\varepsilon)}{f_n(\varepsilon)} = \frac{f(\varepsilon) - \sum^{n} f_n(\varepsilon)}{f_n(\varepsilon)} \to 0 \qquad \text{as } \varepsilon \to 0$$

the remainder R_n is smaller than the last term f_n.

- If the sum has this asymptotic property, one writes

$$f(\varepsilon) \sim \sum^{N} f_n(\varepsilon) \qquad \text{as } \varepsilon \to 0$$

More comments on convergence and divergence

Let's look at Taylor series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + o\left((x - x_0)^2\right)$$

- the sum converges to $f(x)$ as n→∞.
- Taylor series are *de facto* asymptotic expansions

There are two limiting processes for an asymptotic expansion, n→∞ and ε → 0.

- An asymptotic expansion provide an accurate approximation as ε → 0 for each n
- many useful expansions diverge as n→∞.

Optimal truncation

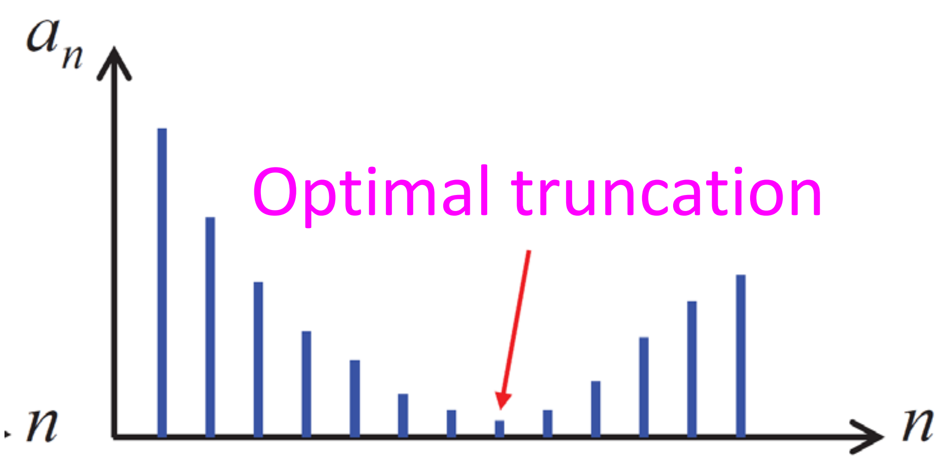
A famous example is the incomplete exponential integral

$$f(x)=\int_x^{\infty} t^{-1}e^{x-t}dt \sim \frac{1}{x}-\frac{1}{x^2}+\frac{2!}{x^3}-\cdots+\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)!\varepsilon^{n+1}}{n!\varepsilon^n}=(n+1)\varepsilon>1 \Rightarrow n_0=\left[\frac{1}{\varepsilon}\right]$$

$$R_{n_0}(\varepsilon)=o\left(a_{n_0}\right)$$

increase as $n>n_0$



3.2.2 Definitions in asymptotic expansion

- In practical problems, it is difficult to calculate enough terms to decide whether the asymptotic series is divergent, as opposed to mathematical asymptotic problem.
- impossible to prove that the remainder after even one or two terms is small enough.
- Luckily, one or two terms is enough for most problems encountered in applied math.

Laplace's method :

- to obtain the asymptotic expansion of certain integrals containing a large parameter.

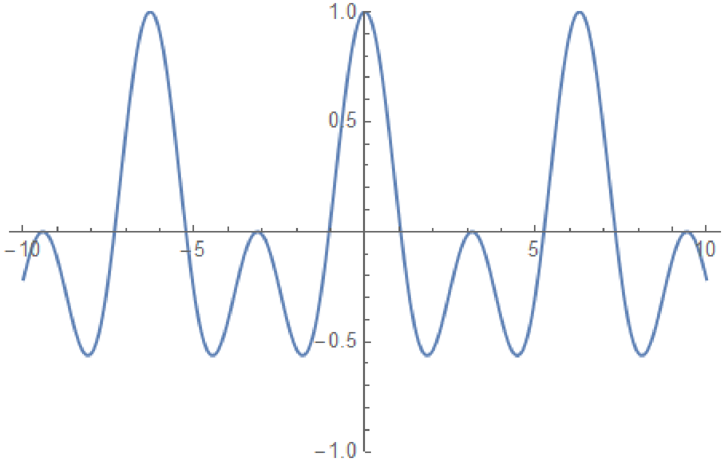
Consider

$$F(\lambda)=\int_{\alpha}^{\beta} g(t)e^{-\lambda f(t)}dt \qquad \lambda >> 1$$

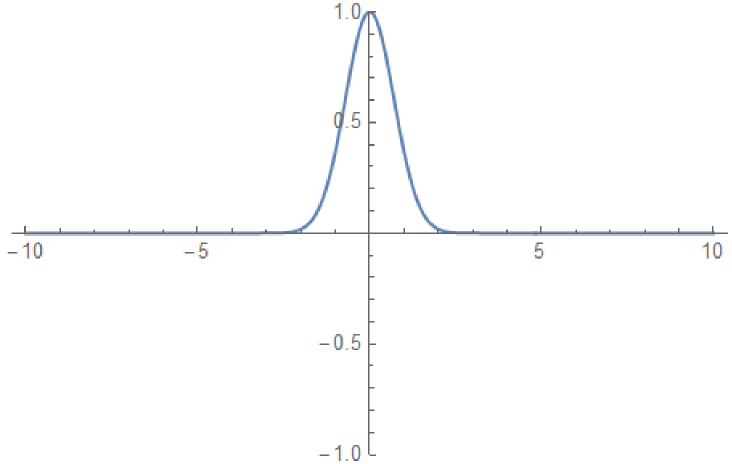
idea:

- the dominant contribution comes from a small portion。

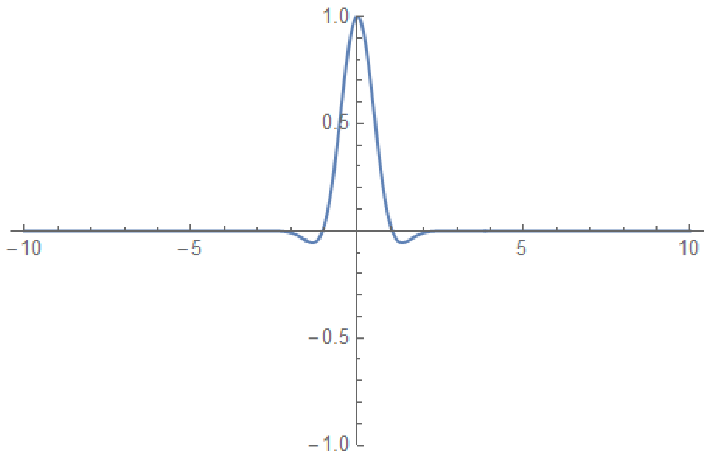
3.2.3 Laplace's method



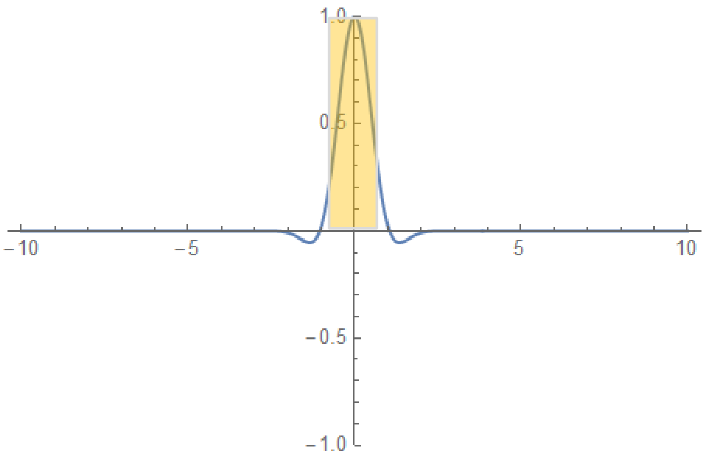
$$g(t) = 0.5(\cos x + \cos 2x)$$



$$f(t) = \exp(-x^2)$$



$$g(t)\,f(t)$$
$$= 0.5(\cos x + \cos 2x)\exp(-x^2)$$



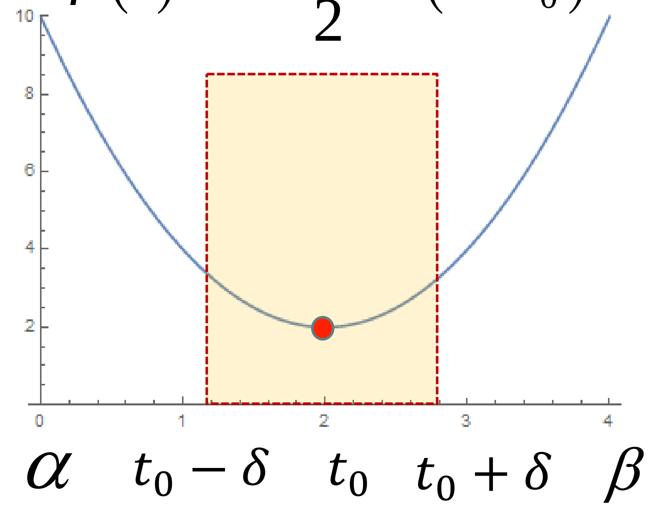
$$F(\lambda) = \int_{\alpha}^{\beta} g(t)\, e^{-\lambda f(t)}\, dt$$

3.2.3 Laplace's method

Let's stare at the integrand

$$F(\lambda) = \int_{\alpha}^{\beta} g(t) e^{-\lambda f(t)} dt$$

$$f(t) \approx \frac{f''(t_0)}{2}(t-t_0)^2$$

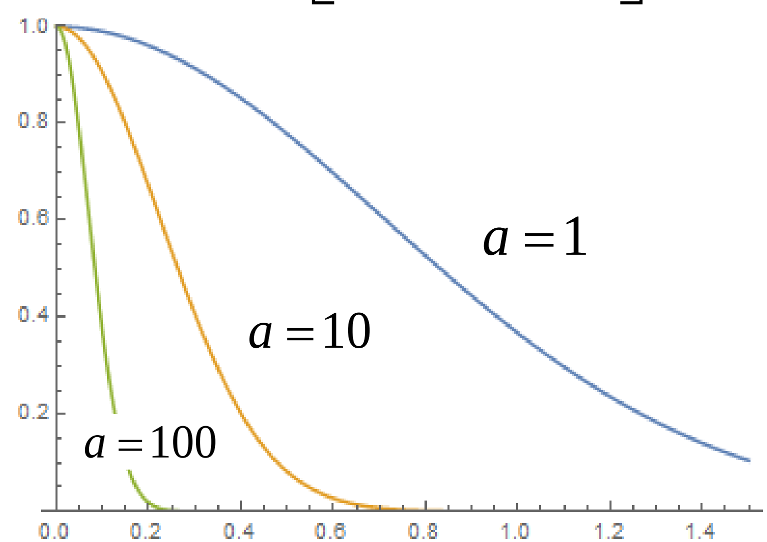


a minimum at t_0

$$F(\lambda) \approx \int_{t_0-\delta}^{t_0+\delta} g(t) e^{-a(t-t_0)^2} dt$$

$$a = \frac{-\lambda f''(t_0)}{2}$$

$$Q(t) = \exp\left[-a(t-t_0)^2\right]$$



Taylor expansion about t_0 .

Assuming $f'(t_0)=0 \quad f''(t_0)>0$

$$f(t)=f(t_0)+f'(t_0)(t-t_0)+\frac{1}{2}f''(t_0)(t-t_0)^2+o\left((t-t_0)^2\right)$$

$$\approx f(t_0)+\frac{1}{2}f''(t_0)(t-t_0)^2$$

thus

$$F(\lambda)=\int_{\alpha}^{\beta} g(t)e^{-\lambda f(t)}dt$$

$$\sim g(t_0)e^{-\lambda f(t_0)}\int_{-\infty}^{\infty}\exp\left[\frac{-\lambda f''(t_0)(t-t_0)^2}{2}\right]dt$$

3.2.3 Laplace's method

$$F(\lambda) = \int_{\alpha}^{\beta} g(t) e^{-\lambda f(t)} dt$$

$$\sim g(t_0) e^{-\lambda f(t_0)} \int_{-\infty}^{\infty} \exp\left[\frac{-\lambda f''(t_0)(t-t_0)^2}{2} \right] dt$$

$$u = (t - t_0)\sqrt{\frac{f''(t_0)}{2}}$$

$$= g(t_0) e^{-\lambda f(t_0)} \left[\frac{2}{f''(t_0)} \right]^{1/2} \int_{-\infty}^{\infty} \exp(-\lambda u^2) du$$

$$= g(t_0) e^{-\lambda f(t_0)} \left[\frac{2\pi}{\lambda f''(t_0)} \right]^{1/2}$$

Gauss integral

$$\int_{-\infty}^{\infty} \exp(-u^2) du = \sqrt{\pi}$$

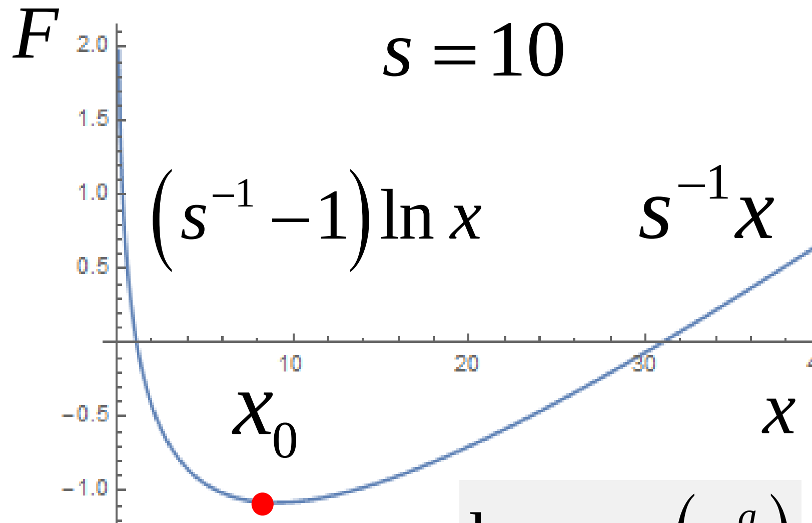
Gamma function $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx \qquad \mathrm{Re}(s)>0$

$$\Gamma\left(\frac{1}{2}\right)=\pi \qquad \Gamma(n)=(n-1)!$$

for s >>1, the gamma function can be reformed as

$$\Gamma(s)=\int_0^\infty e^{(s-1)\ln x}e^{-x}dx$$

$$=\int_0^\infty e^{-sF}dx$$



$$\ln x = o\left(x^a\right)$$

where

$$F(x,s)=\left(s^{-1}-1\right)\ln x+s^{-1}x$$

$$\frac{\partial F}{\partial x}=0 \Rightarrow x_0=s-1$$

The minimal $x_0 = s - 1$ shifts with s. Let $t = \frac{x}{s-1}$

So that the minimal point $x_0 = s - 1$ corresponds to $t_0 = 1$. This leads to

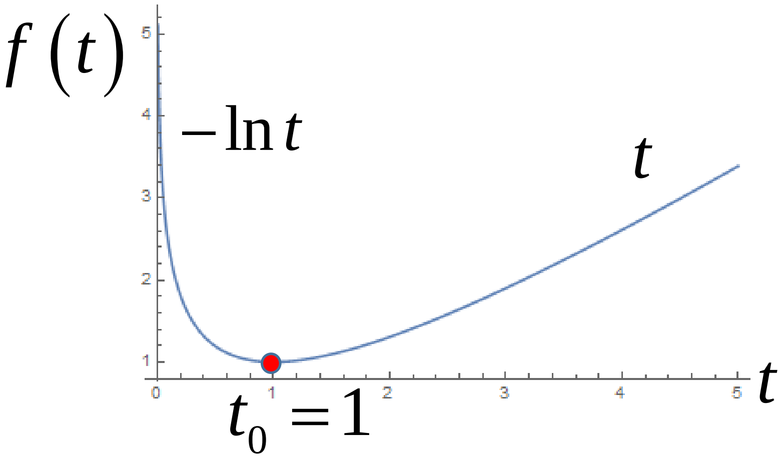
$$\Gamma(s) = \int_0^\infty e^{-sF} dx = (s-1)^s J(\lambda)$$

where

$$J(\lambda) = \int_0^\infty e^{-\lambda f(t)} dt$$

$$f(t) = t - \ln t$$

$$\lambda = s - 1 \gg 1$$



$$\left.\frac{df}{dt}\right|_{t_0=1} = 0 \qquad f(t_0) = 1$$

In order to use Laplace's method, we modify the integrand

$$J=\int_0^\infty e^{-\lambda f(t)}dt=\int_0^\infty e^{-\lambda\left[f(t_0)+f(t)-f(t_0)\right]}dt$$

$$=e^{-\lambda f(t_0)}\int_0^\infty e^{-\lambda w^2(t)}dt$$

where

$$w^2(t)=f(t)-f(t_0)=t-\ln t-1$$

define $w(t)$ as continuous and monotonic

$$\begin{cases} w(t) = -\sqrt{f(t) - f(t_0)} & 0 \le t \le t_0 \\ w(t) = \sqrt{f(t) - f(t_0)} & t > t_0 \end{cases}$$

$$w \in (-\infty, \infty) \quad \leftrightarrow \quad t \in (0, \infty)$$

$$F(\lambda) = \int_{\alpha}^{\beta} g(t) e^{-\lambda f(t)} dt$$

Rewrite

$$J = e^{-\lambda f(t_0)} \int_{-\infty}^{\infty} e^{-\lambda w^2(t)} \frac{dt}{dw} dw$$

$$\frac{dt}{dw} = ?$$

We now try to expand w(t) around $t_0=1$

$$w^2(t) = t - 1 - \ln t$$

$$= (t-1) - \ln\left[1 + (t-1)\right]$$

$$= (t-1) - \left[(t-1) - \frac{1}{2}(t-1)^2 + \frac{1}{3}(t-1)^3 + O\left[(t-1)^4\right] \ldots\right]$$

$$= \frac{1}{2}(t-1)^2 - \frac{1}{3}(t-1)^3 + O\left[(t-1)^4\right]$$

w: a small number

Using the first order

$$w^2(t) = \frac{1}{2}(t-1)^2 - \frac{1}{3}(t-1)^3 + O\left[(t-1)^4\right]$$

$$w^2(t) = \frac{1}{2}(t-1)^2 \Rightarrow t-1 = \sqrt{2}w$$

assume an asymptotic series

$$t-1 = u_0 + u_1 w + u_2 w^2 + \ldots$$

$$\Rightarrow \quad u_0 = 0, \qquad u_1 = \sqrt{2}$$

rewrite $$t-1=\sqrt{2}w\left(1+a_1 w+a_2 w^2 \ldots\right)$$

substituting into the Taylor expansion

$$w^2(t)=\frac{1}{2}(t-1)^2-\frac{1}{3}(t-1)^3+o\left((t-1)^3\right)$$

comparing term by term, we have

$$a_1=\frac{\sqrt{2}}{3},\quad a_2=\frac{1}{18},\quad a_3=-\frac{11\sqrt{2}}{27},\ \ldots.$$

$$b_0=\sqrt{2}$$

Then $$\frac{dt}{dw}=\frac{d}{dw}\left[\sqrt{2}w\left(1+a_1 w+a_2 w^2 \ldots\right)\right]=\sum_{m=0}^{\infty} b_m w^m$$

If term-by-term integration is valid,

$$\frac{dt}{dw} = \sum_{m=0}^{\infty} b_m w^m$$

$$J(\lambda) = e^{-\lambda f(t_0)} \int_{-\infty}^{\infty} e^{-\lambda w^2(t)} \frac{dt}{dw} dw = e^{-\lambda f(t_0)} \sum_{m=0} b_m I_m(\lambda)$$

where

$$I_m(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda w^2} w^m dw$$

$$I_m(\lambda) = 0, \quad m \text{ is odd}$$

$$I_m(\lambda) = \frac{(m-1)!!}{2^{m/2} \lambda^{m/2}} \sqrt{\frac{\pi}{\lambda}}, \quad m \text{ is even}$$

As m is odd

$$I_1(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda w^2} w dw = \frac{-1}{2\lambda} e^{-\lambda w^2} \Big|_{-\infty}^{\infty} = 0$$

$$I_3 = \int_{-\infty}^{\infty} e^{-\lambda w^2} w^3 dw = -\frac{d}{d\lambda} I_1 = 0$$

$$I_5 = \int_{-\infty}^{\infty} e^{-\lambda w^2} w^5 dw = -\frac{d}{d\lambda} I_3 = 0$$

$$I_m = 0$$

As m is even

$$I_0 = \int_{-\infty}^{\infty} e^{-\lambda w^2} dw = \sqrt{\pi / \lambda}$$

$$I_2 = \int_{-\infty}^{\infty} e^{-\lambda w^2} w^2 dw = -\frac{d}{d\lambda} I_0$$

$$I_4 = \int_{-\infty}^{\infty} e^{-\lambda w^2} w^4 dw = -\frac{d}{d\lambda} I_2 = (-1)^2 \frac{d^2}{d\lambda^2} I_0$$

$$I_m = (-1)\frac{dI_{m-2}}{d\lambda} = (-1)^{m/2} \frac{d^{m/2} I_0}{d\lambda^{m/2}} = \frac{(m-1)!!}{2^{m/2} \lambda^{m/2}} \sqrt{\frac{\pi}{\lambda}}$$

To the first order approximation

$$J(\lambda) = e^{-\lambda f(t_0)} \sum_{m=0} b_m I_m(\lambda) \approx e^{-\lambda f(t_0)} \left[b_0 I_0(\lambda) + O\left(\lambda^{-3/2}\right) \right]$$

$$= e^{-\lambda} \sqrt{2\pi / \lambda} \left[1 + O\left(\lambda^{-1}\right) \right]$$

$$b_0 = \sqrt{2} \qquad I_0 = \sqrt{\pi / \lambda}$$

$$\ln \Gamma(s) = \ln (s-1)^s J(\lambda)$$

$$\Gamma(s) = (s-1)^s J(\lambda)$$

$$s = \lambda + 1$$

$$= s \ln(s-1) + \ln e^{-(s-1)} \sqrt{\frac{2\pi}{s-1}} \left[1 + O\left(\lambda^{-1}\right) \right]$$

$$= s \ln(s-1) - (s-1) + \ln \sqrt{\frac{2\pi}{s-1}} + \ln \left[1 + O\left(\lambda^{-1}\right) \right]$$

$$= \left(s - \frac{1}{2} \right) \ln(s-1) - (s-1) + \frac{1}{2} \ln 2\pi + O\left(\frac{1}{s-1} \right)$$

3.3.4 Asymptotic Stirling series for Gamma function

$$\ln\Gamma(s) = \left(s - \frac{1}{2}\right)\ln(s-1) - (s-1) + \frac{1}{2}\ln 2\pi + O\left(\frac{1}{s-1}\right)$$

In the case of integer, i.e. $s = n + 1$

$$\ln n! = \ln\Gamma(n+1)$$

$$= \left(n + \frac{1}{2}\right)\ln n - n + \frac{1}{2}\ln 2\pi + O\left(n^{-1}\right)$$

$$\Gamma(n+1) = n!$$

Stirling's approximation □